MATHEMATICS AS A TOOL :

APPORTIONMENT AND VOTING

LEARNING OUTCOMES:

In this lesson, the students are expected to:

- Solve apportionment problems involving the different method.
- Differentiate : Hamilton apportionment method, Jefferson apportionment method and Huntington – hill method.
- Conduct voting using different methods.

LESSON 1 : APPORTIONMENT

- Is a method of dividing a whole into various parts.
- The process originated in 1790 in the U.S. congress.
- They want to established or select a fairly number of representatives of each state based on state population in the U.S. congressional seat.
- It means that the numbers of representative (the seat) is proportion to the population size being represented.
- Sometimes called " the equal proportion ".
- Different plans were introduce to select right numbers of representative .
- has expanded into different applications in the modern world in economics, accounting , business, law , etc.

The Hamilton Plan

- Early apportionment method used in US congress was introduced by Alexander Hamilton
- Given the number of seat in the US congress will be apportioned between states proportionally to their population.
- Standard divisor (D) –the number of voters represented by each representative.
- Standard quota (Q) – the whole part of the quotient when the population of the sub – group is divided by the standard divisor.

Formulas: Standard Divisor (D) $= \frac{total\ population}{number\ of\ representative}$

$$D = \frac{N}{R}$$

Standard quota (Q) $= \frac{sub-group\ population}{standard\ divisor}$

$$Q = \frac{n}{D}$$

Note : 1. Standard quota , Q must be an integer. In case of decimals, just drop the decimal values .
2. When the total standard quota is not equal to given total apportionedor the number of representative , place an additional representative to the next the sub – group representative with the highest decimal value until the representatives are complete.

Example:

A new school offering the complete six grades in high school has the following enrollment in the different grades below. The administration are to apportioned the 20 teachers for each grade. Calculate

a. The standard divisor
b. The standard quota

Grades	**students**
Grade 12	40
Grade 11	35
Grade 10	22
Grade 9	38
Grade 8	25
Grade 7	39
Total	**199**

Solution : a. Standard Divisor , D $= \frac{N}{R} = \frac{199}{20}$ = **9.95**

b. grade 12 Standard quota , $Q_{12} = \frac{n}{D} = \frac{40}{9.95} = 4.02$

grade 11 Standard quota , $Q_{11} = \frac{n}{D} = \frac{35}{9.95} = 3.51$

grade 10 Standard quota , $Q_{10} = \frac{n}{D} = \frac{22}{9.95} = 2.21$

Grade	student	Q = $\frac{n}{D}$	Q	Q	Corrected no. of teachers
Grade 12	40	$\frac{40}{9.95}$	4.02	4	4
Grade 11	35	$\frac{35}{9.95}$	3.51	3	4
Grade 10	22	$\frac{22}{9.95}$	2.21	2	2
Grade 9	38	$\frac{38}{9.95}$	3.81	3	4
Grade 8	25	$\frac{25}{9.95}$	2.51	2	2
Grade 7	39	$\frac{39}{9.95}$	3.91	3	4
Total :	**199**			17	**20**

The Jefferson Plan

- Another apportionment method used in US congress was introduced by Thomas Jefferson
- This method uses a modified standard divisor the arrives at the correct or exact numbers of representative using trial and error .
- The modified uses an assume value always smaller than the standard divisor.

Example :

A new school offering the complete six grades in high school has the following enrollment in the different grades below. The administration are to apportioned he 20 teachers for each grade. Calculate the number of teachers for each grade using the Jefferson apportionment method.

Grades	students
Grade 12	40
Grade 11	35
Grade 10	22
Grade 9	38
Grade 8	25
Grade 7	39
Total	**199**

Solution :

a. **assume** values for Modified Standard divisor (D_m)

say , D_m = 9

b. by trial and error, solve the number of teachers

grade 12 Standard quota , $Q_{12} = \frac{n}{D} = \frac{40}{9}$ = 4.44

grade 11 Standard quota , $Q_{11} = \frac{n}{D} = \frac{35}{9}$ = 3.88

grade 10 Standard quota , $Q_{10} = \frac{n}{D} = \frac{22}{9}$ = 2.44

note: grade 12 supposedly must have 4 teachers.

c. Another assumption and trial.

say ,D_m = 8.7

grade 12 Standard quota , $Q_{12} = \frac{n}{D} = \frac{40}{8.7}$ = 4.59

grade 11 Standard quota , $Q_{11} = \frac{n}{D} = \frac{35}{8.7}$ = 4.02

grade 10 Standard quota , $Q_{10} = \frac{n}{D} = \frac{22}{8.7}$ = 2.52

Grade	**student**	**Q = $\frac{n}{D}$**	**Q**	**correct no. of teachers**
Grade 12	40	$\frac{40}{8.7}$	4.59	4
Grade 11	35	$\frac{35}{8.7}$	4.02	4
Grade 10	22	$\frac{22}{8.7}$	2.52	2
Grade 9	38	$\frac{38}{8.7}$	4.36	4
Grade 8	25	$\frac{25}{8.7}$	2.87	2
Grade 7	39	$\frac{39}{8.7}$	4.48	4
Total :	**199**			**20**

Analysis :

The above problem , the number of apportionment is exact using the modified standard divisor.

Apportionment principle

- A new representative is added to a sub – group due to an increase in population.
- The representative is assigned to the group in such a way it gives the smallest relative unfairness of apportionment.

Formula : $\mathbf{R} = \frac{A}{C}$

Where : R = relative unfairness of apportionment
A = absolute unfairness of apportionment = $/ C_1 - C_2 /$
C = average population of the sub – group receiving the new Representative.

$$C = \frac{sub-group}{no.of\ representative}$$

Example:

RBSN company wants to add a new call center agent in one of its office. Report indicate an increase in the daily calls of the offices in the past month. Determine which office should get the additional agent. Use the apportionment Principle to justify your answer.

Office branch	Number of agents	Ave. no. of call / day
Makati	62	882
Ortigas	48	996

Solution

Office branch	C_1	C_2	A = / $C_1 - C_2$ /	R = A/C
Makati	$\frac{882}{63} = 14.00$	$\frac{996}{48} = 20.75$	6.75	**0.48**
Ortigas	$\frac{882}{62} = 14.22$	$\frac{996}{49} = 20.32$	6.10	**0.30**

Answer. R = 0.30 (the lowest) means the new agent or representative will goes to Ortigas office.

Analysis.

1. The additional one in C_1 and C_2 indicates that we assume that office or state receives the additional agent or representative.
2. C_1 and C_2 with no additional indicates that office or state does not receives the additional agent or representative.
3. Repeat same procedure when adding another new representative is apportioned.

Huntington – Hill apportioned method

- The present method of apportionment being use by the US congress
- The method that make use of equal proportion.
- The new additional representative to a sub – group must have the highest Huntington number.

Formula : $$H = \frac{(P_A)^2}{a\,(a+1)}$$

Where : P_A = population of the sub – group
a = the current number of representative of sub – group A
H = Huntington – Hill number

Example
RBSN company wants to add a new call center agent in one of its office. Report indicate an increase in the daily calls of the offices in the past month. Determine which office should get the additional agent. Use the Huntington – Hill apportionment to justify your answer.

Office branch	Number of agents	Ave. no. of call / day
Makati	62	882
Ortigas	48	996

Solution

$$H = \frac{(P_A)^2}{a\,(a+1)} = \frac{882^2}{62\,(62+1)} = 199.16$$

$$H = \frac{(P_A)^2}{a\,(a+1)} = \frac{996^2}{48\,(48+1)} = 421.77$$

- Since **H = 421.77** has the highest Huntington – Hill number, So that the Ortigas office will get the new agent.

LESSON 2: VOTING METHODS

1. Plurality Method

- Each voter selects one candidate or choice.
- The winner is the candidate or choice with the most votes.

Example 1: If we vote on a coffee shop for the kind of drinks and 14 people vote for black coffee, 9 for mocha frappe, and 10 people for tea, then the winner of the plurality election is black coffee.

> **Note:**Black coffee did not win by a majority, because it has fewer than 50% of the votes. Black coffee won by a plurality.

Example 2: (Minimum Votes Needed to Win) There are 100 voters in plurality elections between Duterte, Roxas, and Poe. After 70 votes have been counted, Duterte has 38 votes, Roxas has 18 votes, and Poe has 14 votes. How many remaining votes does Duterte need to guarantee he wins?

How to solve this: First, pick your candidate's biggest competition, in this case Roxas. Pretend all 30 votes go to Roxas and Duterte. Let x be the number of votes Duterte needs to tie Roxas in this scenario. Then Roxas gets $30 – x$of the remaining votes. Since it's a tie, $38 + x = 18 + (30 – x)$. Solve for x to get $x = 5.$ If Duterte gets more votes than this, he is guaranteed to win, and so the answer is the smallest number bigger than x, in this case 5 votes.

Strategic Voting: When a person votes in a way that does not reflect his or her true preferences in an attempt to improve the outcome of the election form that person's point of view, it is called *strategic voting*.

Runoff Election:

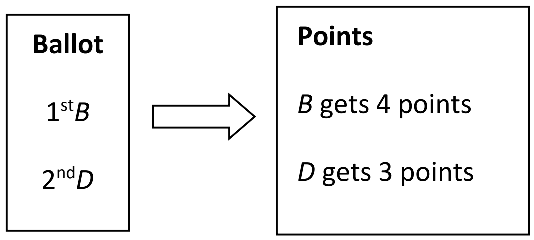
- First a plurality vote is taken.
- If one candidate has more than 50% of the vote, that candidate wins.
- If no candidate has a majority of the votes, a second plurality election is held with a designated number of the top candidates.
- This process repeats until one candidate has more than 50% of the votes.

Example 3: In May 2016 Presidential elections, Duterte with 16, 601, 997 votes, Roxas with 9, 978, 175 votes, and Poe with 9, 100, 991 votes. If there had been a runoff between Duterte and Roxas, what percentage of Poe's supporters would have needed to vote for Duterte for him to have a majority of the vote?

How to solve this: First, adding up the votes from all three candidates, we see that there are 35, 681, 163 votes total. A majority is 1 more than half of this, so 1+ $\frac{1}{2}$(35, 681, 163) = 17, 840, 582.5 or 17, 840, 583. Duterte already has 16, 601, 997 votes so he needs 17, 840, 583 – 16, 601, 997 = 1, 238, 586. This is $\frac{1{,}238{,}586}{9{,}100{,}991} \cdot 100\% = 13.61\%$ of Poe's votes.

2. Borda Count Method

- *Award points* to candidates based on preference schedule, then declare the winner to be the candidate with the most points.



In general, if *N* is the number of candidates…

- Each first-place vote is worth *N* points.
- Each second-place vote is worth *N – 1*points.
- Each third-place vote is worth *N – 2* points.
- …
- Each *N*th-place (i.e., last place) vote is worth 1 point.

Whichever candidate receives the most points wins the election.

Example 1: A corporation would like to invite a new investor. The possibilities are Ayala (A), Bonifacio (B), Calixto (C), and Dancel (D).

All investors of the said corporation are polled. The results:

Number of Voters	**35**	**30**	**20**	**15**
1st choice	B	D	C	D
2nd choice	C	A	A	C
3rd choice	A	B	B	A
4th choice	D	C	D	B

Who is the winner under the Borda Count?

How to solve this:

A Borda Count Election

Number of Voters	**35**	**30**	**20**	**15**
1st choice (4 points)	B: 140	D: 120	C: 80	D: 60
2nd choice (3 points)	C: 105	A: 90	A: 60	C: 45
3rd choice (2 points)	A: 70	B: 60	B: 40	A: 30
4th choice (1 point)	D: 35	C: 30	D: 20	B: 15

Ayala (A): 70 + 90 + 60 + 30 = 250 points

Bonifacio (B): 140 + 60 + 40 + 15 = 255 points

Calixto (C): 70 + 30 + 80 + 45 = 225 points

Dancel (D): 35 + 120 + 20 + 60 = 235 points

Bonifacio wins.

Example 2: A Borda Count Election is held between Theo, Nicole, and Rafael.

Number of Voters	**10**	**8**	**7**	**5**
1st choice	Theo	Nicole	Nicole	Rafael
2nd choice	Nicole	Rafael	Theo	Nicole
3rd choice	Rafael	Theo	Rafael	Theo

How to solve this:

Number of Voters	**10**	**8**	**7**	**5**
1st choice (3 points)	Theo: 30	Nicole: 24	Nicole: 21	Rafael: 15
2nd choice (2 points)	Nicole: 20	Rafael: 16	Theo: 14	Nicole: 10
3rd choice (1 point)	Rafael: 10	Theo: 8	Rafael: 7	Theo: 5

Theo: 30 + 8 + 14 + 5 = 57 points

Nicole: 20 + 24 + 21 + 10 = 75 points

Rafael: 10 + 16 + 7 + 15 = 48 points

Nicole is the winner.

Example 3:

Contestant	Rankings				
A	5	4	1	1	3
B	4	1	5	2	2
C	3	5	4	3	1
D	2	2	2	4	5
E	1	3	3	5	4
Number of Voters	**120**	**90**	**56**	**123**	**31**

How to solve this:

Contestant	Rankings				
A	5 (1 point)	4 (2 points)	1 (5 points)	1 (5 points)	3 (3 points)
B	4 (2 points)	1 (5 points)	5 (1 point)	2 (4 points)	2 (4 points)
C	3 (3 points)	5 (1 point)	4 (2 points)	3 (3 points)	1 (5 points)
D	2 (4 points)	2 (4 points)	2 (4 points)	4 (2 points)	5 (1 point)
E	1 (5 points)	3 (3 points)	3 (3 points)	5 (1 point)	4 (2 points)
Number of Voters	**120**	**90**	**56**	**123**	**31**

Contestant A: 120(1) + 90(2) + 56(5) + 123(5) + 31(3) = 1288 points

Contestant B: 120(2) + 90(5) + 56(1) + 123(4) + 31(4) = 1362 points

Contestant C: 120(3) + 90(1) + 56(2) + 123(3) + 31(5) = 1086 points

Contestant D: 120(4) + 90(4) + 56(4) + 123(2) + 31(1) = 1341 points

Contestant E: 120(5) + 90(3) + 56(3) + 123(1) + 31(2) = 953 points

Thus, using the Borda Count method of voting, contestant B wins.

3. Plurality by Elimination

- The plurality with elimination voting method is also known as an instant run-off voting and sequential run-off voting.
- It is a preferential voting method and candidates that have the least first place votes get eliminated until one candidate has majority of first place votes.

Example 1: Let's say we have a town of 20,000 people electing a mayor using the Plurality with Elimination Voting Method. There are 4 candidates, candidate A, candidate B, candidate C, and candidate D. The results of the election are shown below in a preference schedule:

Number of Voters	**4000**	**7000**	**3000**	**6000**
1st	D	C	A	B
2nd	A	B	B	D
3rd	B	D	C	C
4th	C	A	D	A

Because no candidate received a majority of first place votes, the candidate with fewest first place votes is eliminated, which is candidate C. The adjusted preference schedule is shown below:

Number of Voters	4000	7000	3000	6000
1st	D	B	A	B
2nd	A	D	B	D
3rd	B	A	D	A

Again, because no candidate has the majority of first place votes, the candidate with the fewest first place vote is eliminated, which is candidate D. The adjusted preference schedule is shown below:

Number of Voters	4000	7000	3000	6000
1st	A	B	A	B
2nd	B	A	B	A

Finally, there is candidate who receives the majority of first place votes, which is candidate B with 13,000 votes. So candidate B wins the election using the Plurality with Elimination Voting Method.

Preference Ranking: A *preference ranking* is a voter's order of preference of the candidates.

Example 2: Three candidates are running for chairman of a institutional organization. The preference ranking of the voters are as follows.

	Number of Voters					
	16	**5**	**10**	**8**	**10**	**7**
Perona	1	1	2	3	2	3
Reyes	2	3	1	1	3	2
Santos	3	2	3	2	1	1

a) Who would win in a plurality election with a runoff between the top 2 finishers?
b) In a plurality election, could the seven voters who ranked Santos first and Reyes second achieve a preferable outcome by voting strategically?

c) In a plurality election with a runoff between the top candidates, could the seven voters who ranked Santos first and Reyes second achieve a preferable outcome by voting strategically?

How to solve this:

a) We saw in the plurality election, the top two candidates are Perona and Reyes. We have to see how Santos' voters will change their votes. According to the preference ranking, 10 will switch to Perona so Perona would have his 21 original votes and 10 more, giving him 31. Reyes would have his original 18 and another 7, giving him 25 votes. Perona would thus still win.

b) Yes, if the 7 voters had voters had voted for Reyes in the plurality election instead of Santos, Reyes would have 25 votes, and so Reyes would have won. These 7 votes prefer Reyes to Perona, and so this is a preferable outcome.

c) First, if the 7 voters switch their votes in the first round of voting, they can't change who ends up in the runoff, since Reyes is already in the runoff. Second, in the runoff, they prefer Reyes but voting him doesn't affect the outcome, since Perona still wins. Thus, the answer is no.

4. Pairwise Comparison Method

- Compare each two candidates head-to-head.
- Award each candidate one point for each head-to-head victory.
- The candidate with the most points wins.

Example 1:

Number of Voters	**18**	**11**	**9**	**5**	**2**
1st choice	A	C	D	B	C
2nd choice	B	D	C	D	B
3rd choice	C	B	B	C	D
4th choice	D	A	A	A	A

- Compare A to B.
  - 18 voters prefer A.
  - 11 + 9 + 5 + 2 = 27 voters prefer to B.
  - B wins the pairwise comparison and gets 1 point.
- Compare A to C.
  - 18 voters prefer A.
  - 11 + 9 + 5 + 2 = 27 voters prefer to C.
  - C wins the pairwise comparison and gets 1 point.
- Compare A to D.
  - 18 voters prefer A.

- 11 + 9 + 5 + 2 = 27 voters prefer to D.
- D wins the pairwise comparison and gets 1 point.

- Compare B to C.
  - 18 + 5 = 23 voters prefer B.
  - 11 + 9 + 2 = 22 voters prefer to C.
  - C wins the pairwise comparison and gets 1 point.

- Compare B to D.
  - 18 + 5 + 2 = 25 voters prefer B.
  - 11 + 9 = 20 voters prefer to D.
  - B wins the pairwise comparison and gets 1 point.

- Compare C to D.
  - 18 + 11 + 3 = 32 voters prefer C.
  - 9 + 5 = 14 voters prefer D.
  - C wins the pairwise comparison and gets 1 point.

Contestant	**Total Number of Points**
A	0
B	2
C	3
D	1

Therefore, C wins using pairwise comparison method.

Example 2: Let's say we have a town of 20,000 people electing a mayor using the Plurality with Elimination Voting Method. There are 4 candidates, candidate A, candidate B, candidate C, and candidate D.

Number of Voters	4000	7500	3000	5500
1st	D	C	A	B
2nd	A	B	B	D
3rd	B	D	C	C
4th	C	A	D	A

- Compare A to B.
  - 4000 + 3000 = 7000 voters prefer A.
  - 7500 + 5500 = 13000 voters prefer to B.
  - B wins the pairwise comparison and gets 1 point.
- Compare A to C.
  - 4000 + 3000 = 7000 voters prefer A.
  - 7500 + 5500 = 13000 voters prefer to C.
  - C wins the pairwise comparison and gets 1 point.
- Compare A to D.
  - 3000 voters prefer A.
  - 4000 + 7500 + 5500 = 17000 voters prefer to D.
  - D wins the pairwise comparison and gets 1 point.
- Compare B to C.
  - 4000 + 3000 + 5500 = 12500 voters prefer B.
  - 7500 voters prefer to C.
  - B wins the pairwise comparison and gets 1 point.
- Compare B to D.
  - 7500 + 3000 + 5500 = 16000 voters prefer B.
  - 4000 voters prefer to D.
  - B wins the pairwise comparison and gets 1 point.
- Compare C to D.
  - 7500 + 3000 = 10500 voters prefer C.
  - 4000 + 5500 = 9500 voters prefer D.
  - C wins the pairwise comparison and gets 1 point.

Contestant	Total Number of Points
A	0

B	3
C	2
D	1

Therefore, B wins using pairwise comparison method.